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On the microscopic realisation of the CM(3) or MQC model for nuclear collective motion†

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Abstract. Transformations to collective coordinates are introduced by considering the action of collective and intrinsic groups. The methods are applied to the microscopic realisation of the CM(3) or MQC model for nuclear collective motion. A special coordinate transformation is found which allows one to discuss the collectivity of observables associated with this realisation.

1. Introduction

In this paper we discuss the separation problem of the 'collective' kinetic energy and velocity field for the system of $A \equiv n+1$ particles. This problem was studied before in the framework of nuclear collective motion theories. The approaches in which we are interested are based on group theoretical treatments. Two different directions are involved.

One of them concerns the problem of the coordinate transformation for the system of A particles such that the collective coordinates which are changed through the collective motion appear explicitly. This concept, to the best of our knowledge, was introduced by Lipkin *et al* (1955), Villars (1957), and also Villars and Cooper (1970). They 'added' to the system of $3A$ cartesian single-particle coordinates three collective coordinates for the collective rotations of the system and introduced simultaneously three holonomic constraints. This idea was improved by Zickendraht (1971) and Dzyublik *et al* (1972). They introduced, besides the rotational coordinates, three coordinates for the deformations and vibrations which generalise the β and γ coordinates of Bohr (1952) and of Bohr and Mottelson (1953). Zickendraht and Dzyublik *et al* used six holonomic constraints in order to get $3n-6$ intrinsic coordinates which Dzyublik *et al* related explicitly to the SO(n) group. This type of collectivity with six collective coordinates we are going to call SO(n) collectivity.

The other idea consists of the study of the phenomenological CM(3) or MQC model for the collective motion (mass quadrupole collective model) (Cusson 1968, Weaver *et al* 1973, 1976, Gulshani and Rowe 1976, Rosensteel and Rowe 1979). These models were based on the idea of spectrum generating algebras (Dothan *et al* 1965). The authors of these papers introduced nine collective degrees of freedom: the six degrees already mentioned for the vibrations and rotations (in the frame of the older CM(3)

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model there were only five, because the monopole mode was not included) and three $\mathscr{SO}(3)$ degrees of freedom which belong to the vortex spin (Weaver *et al* 1973) and yield the collective flow which is not irrotational. The collective coordinates in the CM(3) model are the $\mathscr{SL}(3, \mathbb{R})$ group parameters and in the MQC model are the $\mathscr{GL}_+(3, \mathbb{R})$ parameters.

Gulshani and Rowe (1976) tried to introduce the MQC model through the coordinate transformation which was similar to that of Villars (1957) and Villars and Cooper (1970) (redundant coordinates), introducing nine $\mathscr{GL}_+(3, \mathbb{R})$ collective coordinates and nine constraints, from which six were holonomic but three non-holonomic, which we are going to discuss later on. Due to these nine constraints they obtained the total separation of the kinetic energy into a $\mathscr{GL}_+(3. \mathbb{R})$ collective part and the intrinsic one, and they also extracted the collective $\mathscr{GL}_+(3, \mathbb{R})$ velocity field which was no longer irrotational.

Other attempts were undertaken in order to obtain a microscopically based MQC model. Rowe and Rosensteel (1979, 1980) and Rowe (1981) introduced the coordinate transformation based on the orbit analysis which has to be viewed as a transformation of momenta rather than coordinates. Buck *et al* (1979) did similar work. All these authors actually obtained the results already found by Zickendraht (1971) and Dzyublik *et al* (1972) but interpreted them differently: they claimed that the total separation of the $\mathscr{GL}_+(3, \mathbb{R})$ collective kinetic energy from the intrinsic one was obtained without making use of non-integrable constraints.

In § 2 of this paper we re-derive the results of Zickendraht (1971), Dzyublik *et al* (1972), Rowe and Rosensteel (1979, 1980) and Buck *et al* (1979) in the frame of an orbit analysis WRT the $\mathscr{SO}(3) \times \mathrm{SO}(n)$ group action. We call the procedure the $\mathscr{SO}(3) \times \mathrm{SO}(n)$ scheme. We claim that this coordinate transformation is good only for the already defined $\mathrm{SO}(n)$ collectivity. In order to give the answer to whether and under what conditions the MQC collective part of the kinetic energy and the velocity field can be totally separated from the intrinsic motion, we introduce in § 3 the coordinate transformation based on the orbit analysis WRT the $\mathscr{GL}_+(3, \mathbb{R}) \times \mathrm{SO}(n)$ group and call it the $\mathscr{GL}_+(3, \mathbb{R}) \times \mathrm{SO}(n)$ scheme.

2. The $\mathscr{SO}(3) \times \mathrm{SO}(n)$ scheme

The configuration of $n+1$ particles in three-dimensional Euclidean space can be viewed as a point in the $3(n+1)$-dimensional Euclidean space $E^{3(n+1)}$. Considering the centre of mass separated, our problem is reduced to E^{3n}. We map E^{3n} on $\mathbb{R}^{3n}$ with the cartesian coordinate chart, where each point of E^{3n} is represented by x_{is}, $i = 1, 2, 3$ is the space index and $s = 1, 2, \ldots, n$ the relative Jacobi vector index. On this space we define the action of the $\mathscr{SO}(3) \times \mathrm{SO}(n)$ group so that the 'collective' group $\mathscr{SO}(3)$ acts from the left on the index i providing the *same* action on all relative vectors. The 'intrinsic' group $\mathrm{SO}(n)$ acts from the right on the index s†

$$x_{is} \to \sum_{i=1}^{3} \sum_{s=1}^{n} {}^{t}\tilde{o}_{li} x_{is} \tilde{o}'_{st} \qquad \tilde{o} \in \mathscr{SO}(3), \tilde{o}' \in \mathrm{SO}(n).$$

The idea of this action is to replace as many cartesian coordinates as possible by group parameters. The three parameters from the $o(\omega) \in \mathscr{SO}(3)$ are the coordinates

† We use script letters for the 'collective' group and latin letters for the 'intrinsic' group.

for the collective rotations. The three coordinates for the deformation and vibration are introduced by the diagonalisation of the collective tensor

$$\begin{aligned}\mathcal{T}_{ij}(x) &:= \sum_{s=1}^{n} x_{is}x_{js} \\ &= \sum_{l=1}^{3} {}^{\mathrm{t}}\!\mathscr{o}_{il}(\omega)\mu_l \mathscr{o}_{lj}(\omega) \qquad \mu_l \geqslant 0.\end{aligned}$$

The collective tensor is related to the instantaneous tensor of inertia by

$$I_{ij} = -m[\mathcal{T}_{ij} - \mathrm{Tr}\,\mathcal{T}\delta_{ij}].$$

The coordinates $\mathring{x}_l := +\sqrt{\mu_l}$, $l = 1, 2, 3$, are related to the instantaneous principal moments as

$$I_1 = m(\mathring{x}_2^2 + \mathring{x}_3^2) \qquad I_2 = m(\mathring{x}_3^2 + \mathring{x}_1^2) \qquad I_3 = m(\mathring{x}_1^2 + \mathring{x}_2^2).$$

As an intermediate result one writes the point x_{is} as

$$x_{is} = \sum_{l=1}^{3} \mathscr{o}_{il}(\omega)\mathring{x}_l \mathscr{o}'_{ls}(\beta) \qquad \mathscr{o} \in \mathcal{SO}(3), \mathscr{o}' \in \mathrm{SO}(n)$$

and one restricts $\mathring{x}_l$ so that $\mathring{x}_1 > \mathring{x}_2 > \mathring{x}_3 > 0$. This restriction is allowed for a quantum mechanical system, but not for the classical system. On the right-hand side one has more than $3n$ coordinates and in order to get rid of superfluous coordinates one searches for the stability group H of the representative point $x_{is} = \mathring{x}_i\delta_{is}$ and finds $\mathrm{H} = \mathrm{diag}(\mathrm{M}\times\mathrm{M})\times\mathrm{SO}(n-3)$, where

$$\mathrm{M} = \left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right\}$$

and $\mathrm{M} < \mathcal{SO}(3)$ and $\mathrm{M} < \mathrm{SO}(n)$, $\mathrm{SO}(n-3) < \mathrm{SO}(n)$. The group $\mathrm{SO}(n-3)$ acts on the relative Jacobi vector indices from 4 to n. The subgroup M is not essential for the number of parameters. One finally writes the point

$$x_{is} = \sum_{l=1}^{3} \mathscr{o}_{il}(\omega)\mathring{x}_l \mathscr{o}'_{ls}(\beta)$$

where $\mathscr{o}(\omega) \in \mathcal{SO}(3)$ $(\in \mathcal{SO}(3)/\mathrm{M}$, left coset space) $\mathring{x}_1 > \mathring{x}_2 > \mathring{x}_3 > 0$ and $\mathscr{o}'(\beta) \in \mathrm{SO}(n)/\mathrm{SO}(n-3)$, right coset space. For the detailed development of an orbit analysis which leads to these results see Rowe and Rosensteel (1980).

Proposition 2.1. The Jacobi matrix for the coordinate transformation

$$\begin{aligned}(x_{ls}|l = 1, 2, 3, s = 1, 2, \ldots, n) &\to (\omega_\mu, \mathring{x}_i, \beta_\sigma|\mu, i = 1, 2, 3, \sigma = 1, 2, \ldots, 3n-6) \\ &\equiv (\zeta_\nu|\nu = 1, 2, \ldots, 3n)\end{aligned}$$

is given through its matrix elements

$$\mathcal{J}_{\nu,ls} := \frac{\partial x_{ls}}{\partial \zeta_\nu}$$

$$\frac{\partial x_{ls}}{\partial \omega_\mu} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathscr{o}_{lj}(\omega)\,{}^{\mathrm{R}}\tilde{\Psi}_{\mu,ji}(\omega)\mathscr{o}'_{is}(\beta) \tag{1}$$

(2) $$\frac{\partial x_{ls}}{\partial \mathring{x}_w} = o_{lw}(\omega) o'_{ws}(\beta)$$

(3) $$\frac{\partial x_{ls}}{\partial \beta_\tau} = \sum_{i=1}^{3} \sum_{t=1}^{n} o_{li}(\omega) \mathring{x}_i {}^{\mathrm{L}}\tilde{\Psi}_{\tau,it}(\beta) o'_{ts}(\beta)$$

where ${}^{\mathrm{R}}\Psi_{\mu,ji}$ is as in (A21)† and ${}^{\mathrm{L}}\tilde{\Psi}_{\tau,it}$ as in (A18).

In order to express the old cartesian quantum mechanical or classical momenta p_{is} after the coordinate transformation we need the inverse Jacobi matrix $\mathscr{J}^{-1}_{is,\nu}$

$$p_{is} = \sum_{\nu=1}^{3n} \mathscr{J}^{-1}_{is,\nu} \pi_\nu.$$

The momenta p_{is} are either $-\mathrm{i}\hbar\, \partial/\partial x_{is}$ or the canonically conjugate momenta to the x_{is}, and π_ν are $-\mathrm{i}\hbar\, \partial/\partial \zeta_\nu$ or canonically conjugate momenta to ζ_ν. If one is interested only in the expression for the kinetic energy in the new coordinates it is of course sufficient to invert the metric tensor $g_{\mu n}$ (Rowe and Rosenteel 1980, Buck *et al* 1979).

Proposition 2.2. The inverse Jacobi matrix $\mathscr{J}^{-1}$ is given by its matrix elements

$$\mathscr{J}^{-1}_{ls,\nu} = \frac{\partial \zeta_\nu}{\partial x_{ls}}$$

(1) $$\frac{\partial \omega_\mu}{\partial x_{ls}} = \sum_{v=1}^{3} \sum_{g=1}^{3} o_{lg}(\omega) \frac{\mathring{x}_v}{\mathring{x}_v^2 - \mathring{x}_g^2} {}^{\mathrm{R}}\tilde{\Theta}_{gv,\mu}(\omega) o'_{vs}(\beta)$$

(2) $$\frac{\partial \mathring{x}_w}{\partial x_{ls}} = o_{lw}(\omega) o'_{ws}(\beta)$$

(3) $$\frac{\partial \beta_\tau}{\partial x_{ls}} = \sum_{v=1}^{3} \sum_{g=1}^{3} o_{lg}(\omega) \frac{\mathring{x}_g}{\mathring{x}_g^2 - \mathring{x}_v^2} {}^{\mathrm{L}}\tilde{\Theta}_{gv,\tau}(\beta) o'_{vs}(\beta) + \sum_{v=4}^{n} \sum_{g=1}^{3} o_{lg}(\omega) \mathring{x}_g^{-1} {}^{\mathrm{L}}\tilde{\Theta}_{gv,\tau}(\beta) o'_{vs}(\beta)$$

where ${}^{\mathrm{R}}\tilde{\Theta}_{gv,\mu}$ and ${}^{\mathrm{L}}\tilde{\Theta}_{gv,\tau}$ are as in (A23).

For the proof the propositions A16 and A17 are essential.
We now calculate the $\mathscr{SO}(3)$ left action and SO(n) right action generators on the new parametrised manifold, making use of (2.2).

Proposition 2.3. The angular momentum operator of the system of $n+1$ particles

$${}^{\mathrm{L}}\hat{\mathscr{X}}^a_{lk} := -\mathrm{i}\hbar\, {}^{\mathrm{L}}\mathscr{X}^a_{lk}$$

where

$${}^{\mathrm{L}}\mathscr{X}^a_{lk} := \sum_{s=1}^{n} \left(x_{ls} \frac{\partial}{\partial x_{ks}} - x_{ks} \frac{\partial}{\partial x_{ls}} \right)$$

acting on the orbit $\mathscr{SO}(3) \times \mathrm{SO}(n)/(\mathrm{M} \times \mathrm{SO}(n-3))$, takes the form

$${}^{\mathrm{L}}\hat{\mathscr{X}}^a_{lk} = \mathrm{i}\hbar \sum_{\mu=1}^{3} {}^{\mathrm{L}}\Theta_{lk,\mu}(\omega) \frac{\partial}{\partial \omega_\mu}.$$

† The symbol (A21) means proposition A21, listed in the appendix.

The $\mathscr{SO}(3)$ left action generator ${}^{\mathrm{L}}\mathscr{X}^a_{lk}$ can also be written in the form

$$
{}^{\mathrm{L}}\mathscr{X}^a_{lk} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{\mathrm{t}}d^{-1}_{lk,gv}(\omega) L_{gv}
$$

where L_{gv} is defined by

$$
L_{gv} := \sum_{\mu=1}^{3} {}^{\mathrm{R}}\Theta_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu}.
$$

In the derivation one makes use of (2.2), (A8) and (A19). Note that the right action generators L_{gv} multiplied by $-\mathrm{i}\hbar$ are not exactly the angular momenta components 'referred' to the body-fixed system (Biedenharn and Brussard 1965) because they fulfil the same commutation relations as the angular momenta in the laboratory frame as the right and left action generators (see (A12)). The operators $-L_{gv} := \mathrm{i}\hbar L_{gv}$ are the momenta 'referred' to the body-fixed system.

Proposition 2.4. The right action generators of SO(n) acting on the orbit $\mathscr{SO}(3)\times \mathrm{SO}(n)/(\mathrm{M}\times\mathrm{SO}(n-3))$, expressed through the new coordinates ω and β, are

$$
{}^{\mathrm{R}}X^a_{st} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{\mathrm{t}}d_{st,gv}(\beta)\mathscr{L}_{gv} - \sum_{g=1}^{3}\sum_{v=4}^{n} {}^{\mathrm{t}}d_{st,gv}(\beta)\mathscr{I}_{gv}
$$

where $\mathscr{L}_{gv}$ and $\mathscr{I}_{gv}$ are defined as

$$
\mathscr{L}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{\mathrm{L}}\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad g<v=1,2,3
$$

$$
\mathscr{I}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{\mathrm{L}}\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad g=1,2,3,\ v=4,5,\ldots,n
$$

and the matrix d is given in (A19). The generators ${}^{\mathrm{R}}X^a_{st}$ can also be written

$$
{}^{\mathrm{R}}X^a_{st} = \sum_{\tau=1}^{3n-6} {}^{\mathrm{R}}\Theta_{st,\tau}(\beta)\frac{\partial}{\partial\beta_\tau}
$$

and the Hermitian operator is ${}^{\mathrm{R}}\hat{X}^a_{st} := -\mathrm{i}\hbar\, {}^{\mathrm{R}}X^a_{st}$.

In the derivation one makes use of (2.2), (A8), (A16) and (A19). Note that $\mathscr{L}_{g<v}$, $g, v=1,2,3$, although defined to include in general all partial derivatives WRT the β (the β parametrise the right coset space $\mathrm{SO}(n)/\mathrm{SO}(n-3)$), through some particular choice of the β can be recognised as the left action generators on an $\mathscr{SO}(3)$ submanifold of the manifold $\mathrm{SO}(n)/\mathrm{SO}(n-3)$:

$$
\frac{\mathrm{SO}(n)}{\mathrm{SO}(n-3)} \simeq \left(\begin{array}{c|c} \mathrm{SO}(3) & 0 \\ \hline 0 & \mathbb{1}_{n-3}\end{array}\right) \times \frac{\mathrm{SO}(n)}{\mathrm{SO}(3)\times\mathrm{SO}(n-3)}.
$$

One can assume that the first three β: β_1, β_2 and β_3 parametrise the SO(3) submanifold. The differential operator multiplied by $-\mathrm{i}\hbar$, $\hat{\mathscr{L}}_{g<v} := = -\mathrm{i}\hbar\mathscr{L}_{g<v}$ is the angular momentum in the body fixed frame (Buck *et al* 1979) known as the vortex spin. The $\mathscr{I}_{gv}$, $g=1,2,3$, $v=4,5,\ldots,n$, by this choice of coordinates still include partial derivatives WRT *all* β including also β_1, β_2 and β_3, the 'collective' vortex coordinates. This property of $\mathscr{I}$ will be discussed in particular in the next section.

The proposition analogous to (2.3) and (2.4) can be stated for the classical momenta (Hamiltonian functions of the Poisson action (Arnold 1978)) replacing the partial derivatives $-\mathrm{i}\hbar\, \partial/\partial\zeta_\nu$ by the momenta π_ν canonically conjugate to the corresponding coordinates ζ_ν.

Proposition 2.5. The quantum mechanical kinetic energy on $\mathbb{R}^{3n}$ is given in the new set of coordinates $(\omega_\mu, \mathring{x}_w, \beta_\tau)$ by

$$
\begin{aligned}
T = {} & \frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\mathring{x}_v^2+\mathring{x}_g^2}{(\mathring{x}_v^2-\mathring{x}_g^2)^2}\hat{L}_{gv}^2+\frac{1}{2m}\sum_{v=1}^{3}\hat{t}_v^2 \\
& -\frac{\mathrm{i}\hbar}{2m}\left(\sum_{v=1}^{3}\frac{n-3}{\mathring{x}_v}\hat{t}_v+\sum_{v=1}^{3}\sum_{g=1}^{3}\frac{2\mathring{x}_v}{\mathring{x}_v^2-\mathring{x}_g^2}\hat{t}_v\right) \\
& +\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\mathring{x}_v^2+\mathring{x}_g^2}{(\mathring{x}_v^2-\mathring{x}_g^2)^2}\hat{\mathscr{L}}_{gv}^2+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=4}^{n}\mathring{x}_g^{-2}\hat{\mathscr{I}}_{gv}^2 \\
& +\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{4\mathring{x}_v\mathring{x}_g}{(\mathring{x}_v^2-\mathring{x}_g^2)^2}\hat{L}_{gv}\hat{\mathscr{L}}_{gv}
\end{aligned}
$$

where all Hermitian differential operators which appear are defined in (2.3) and (2.4) except for the vibrational operators $\hat{t}_v := -\mathrm{i}\hbar\,\partial/\partial\mathring{x}_v$, $v = 1, 2, 3$.

In deriving (2.5) one makes use essentially of (2.4),

$$T=\frac{1}{2m}\sum_{\nu=1}^{3n}\sum_{\mu=1}^{3n}\sum_{i=1}^{3}\sum_{s=1}^{n}\mathscr{I}^{-1}_{is,\nu}\frac{\partial}{\partial\zeta_\nu}\mathscr{I}^{-1}_{is,\mu}\frac{\partial}{\partial\zeta_\mu}$$

and also of (A9), (A22), (A17), (A18), (2.3) and (2.4). The interpretation of the expression for the kinetic energy given in (2.5) is as follows. With respect to the SO(n) collectivity the first three terms are collective, the fourth and the fifth are intrinsic and the term proportional to $\hat{L}\hat{\mathscr{L}}$ is the coupling term. With respect to the $\mathscr{GL}_+(3,\mathbb{R})$ collectivity, with the special, already mentioned, choice of the β_τ coordinates, it can be found that the first five terms in the expression of T in (2.5) are purely collective, but, as we also mentioned, the $3n-9$ differential operators $\mathscr{I}$ are defined as to contain partial derivatives WRT all $3n-6$ SO(n)/SO($n-3$) right coset space parameters and cannot be considered as intrinsic. In order to see if the complete separation of the $\mathscr{GL}_+(3,\mathbb{R})$ collective kinetic energy from the pure intrinsic part can be achieved and under what conditions, it is discussed in § 3 with the use of a special coordinate transformation. This provides us with intrinsic differential operators which include only partial derivatives WRT the intrinsic coordinates.

An analogous expression for the classical kinetic energy can easily be determined using

$$T_{\mathrm{cl}}=\frac{1}{2m}\sum_{i=1}^{3}\sum_{s=1}^{n}\sum_{\nu=1}^{3n}\sum_{\mu=1}^{3n}\mathscr{I}^{-1}_{is,\nu}\pi_\nu\mathscr{I}^{-1}_{is,\mu}\pi_\mu$$

where π_ν and π_μ are the classical momenta. The result is completely analogous to the quantum mechanical one, only the linear term in momenta does not appear.

Now we try to extract the SO(n) linear (collective) velocity field which should correspond to an irrotational flow. We proceed as in Gulshani and Rowe (1976). We first calculate p_{is} (the momenta conjugate to the coordinates x_{is}) as a function of the new coordinates and momenta:

$$
\begin{aligned}
p_{is} &= \sum_{\nu=1}^{3n} \mathscr{I}^{-1}_{is,\nu}\pi_\nu \\
&= \sum_{j=1}^{3}\sum_{g=1}^{3}\sum_{v=g+1}^{3} (o_{ig}(\omega)o_{jv}(\omega)+o_{iv}(\omega)o_{jg}(\omega))\frac{1}{\mathring{x}_v^2-\mathring{x}_g^2}L_{gv}x_{js} \\
&\quad + \sum_{j=1}^{3}\sum_{w=1}^{3} o_{iw}(\omega)o_{jw}(\omega)\mathring{x}_w^{-1}\pi_w x_{js} \\
&\quad + \sum_{j=1}^{3}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(o_{ig}(\omega)\frac{\mathring{x}_g}{\mathring{x}_v}o_{jv}(\omega)+o_{iv}(\omega)\frac{\mathring{x}_v}{\mathring{x}_g}o_{jg}(\omega)\right)\frac{1}{\mathring{x}_v^2-\mathring{x}_g^2}\mathscr{L}_{gv}x_{js} \\
&\quad - \sum_{g=1}^{3}\sum_{v=4}^{n} o_{ig}(\omega)\mathring{x}_g^{-1}o'_{vs}(\beta)\mathscr{I}_{gv}
\end{aligned}
$$

where the classical momenta are defined as follows:

$$
L_{gv} := \sum_{\mu=1}^{3} {}^{\mathrm{R}}\Theta_{gv,\mu}(\omega)\pi_\mu \qquad g<v=1,2,3
$$

$$
\mathscr{L}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{\mathrm{L}}\Theta_{gv,\tau}(\beta)\pi_\tau \qquad g<v=1,2,3
$$

$$
\mathscr{I}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{\mathrm{L}}\Theta_{gv,\tau}(\beta)\pi_\tau \qquad g=1,2,3,\ v=4,5,\ldots,n
$$

and π_w are the momenta canonically conjugate to the coordinate $\mathring{x}_w$, $w=1,2,3$. With respect to SO(n) collectivity, the first two terms in the expression for p_{is} determine the collective part. For the local Hamiltonian we have

$$
m\boldsymbol{v}_s = \boldsymbol{p}_s
$$

and we extract the collective part of the velocity $\boldsymbol{v}_s$

$$
v_{is}^{\mathrm{coll}} = \frac{1}{m}\sum_{j=1}^{3}(\chi_{ij}^{\mathrm{vib}}+\chi_{ij}^{S})x_{js}
$$

where

$$
\chi_{ij}^{\mathrm{vib}} := \sum_{w=1}^{3} o_{iw}(\omega)o_{jw}(\omega)\mathring{x}_w^{-1}\pi_w
$$

$$
\chi_{ij}^{\mathrm{vib}} = \chi_{ji}^{\mathrm{vib}}
$$

$$
\chi_{ij}^{S} := \sum_{g=1}^{3}\sum_{v=g+1}^{3}(o_{ig}(\omega)o_{jv}(\omega)+o_{iv}(\omega)o_{jg}(\omega))\frac{1}{\mathring{x}_v^2-\mathring{x}_g^2}L_{gv}
$$

$$
\chi_{ij}^{S} = \chi_{ji}^{S}.
$$

We define the velocity field $\boldsymbol{v}$ at the point $\boldsymbol{x}$ to be the collective component of the velocity $\boldsymbol{v}_s$ for the Jacobi vectors s at $\boldsymbol{x}_s \equiv \boldsymbol{x}$:

$$
v_i = \frac{1}{m}\sum_{j=1}^{3}(\chi_{ij}^{\mathrm{vib}}+\chi_{ij}^{S})x_j \qquad i=1,2,3.
$$

It is clear that the SO(n) (collective) velocity field separates completely from the intrinsic part and is irrotational, rot $\boldsymbol{v}=0$.

The first three terms of p_{is} are the 'candidates' for the (collective) velocity field WRT the $\mathscr{GL}_+(3, \mathbb{R})$ collectivity: the momenta $\mathscr{L}_{gv}$ can still be considered as being related to the SO(3) submanifold of the right coset space $SO(n)/SO(n-3) \equiv C^R$, but the part proportional to $\mathscr{I}$ is related to the whole C^R and cannot be defined as purely intrinsic WRT $\mathscr{GL}_+(3, \mathbb{R})$ collectivity. It should be stressed again that all the classical analysis in this section was done on a phase space related to special orbits in configuration space defined by the condition $\mathring{x}_1 > \mathring{x}_2 > \mathring{x}_3 > 0$. Whereas the other types of orbits in the quantum mechanical case are of measure zero, in the classical case one should also consider all other types of orbits. Similar remarks also apply to the next section (see (3.1.3)).

3. The $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ scheme

As already stressed in the introduction and in § 2, in order to study the problem of the separation of the $\mathscr{GL}_+(3, \mathbb{R})$ collective kinetic energy and the velocity field we introduce some set of coordinates where the nine collective coordinates are defined as $\mathscr{GL}_+(3, \mathbb{R})$ parameters and the additional $3n-9$ coordinates of the right coset space $SO(n)/(SO(3) \times SO(n-3)) \equiv C^R$ are intrinsic. We introduce the collective momenta as the vector fields on the $\mathscr{GL}_+(3, \mathbb{R})$ manifold and corresponding intrinsic momenta on the right coset space C^R. All this we achieve through the coordinate transformation based on the $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ group action on $\mathbb{R}^{3n}$ and a corresponding orbit analysis. This procedure we call the $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ scheme.

3.1. The $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ scheme, orbit analysis

Definition 3.1.1. The action of $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ group on $\mathbb{R}^{3n}$ is introduced as the left action of $\mathscr{GL}_+(3, \mathbb{R})$ and the right action of SO(n) on the points of $\mathbb{R}^{3n}$ which are the Cartesian coordinates of n Jacobi vectors ($x_{is}|i = 1, 2, 3,\ s = 1, 2, \ldots, n$) for the system of $n+1$ particles:

$$\sum_{t=1}^{n} \sum_{l=1}^{3} (\tilde{x}')^{-1}_{il} x_{lt} \tilde{o}''_{ts}(\gamma)$$

where $\tilde{x}' \in \mathscr{GL}_+(3, \mathbb{R})$, $\tilde{o}''(\gamma) \in SO(n)$.

We choose the representative point to be $\mathring{x}_{is} = \delta_{is}$, $i = 1, 2, 3$, $s = 1, 2, \ldots, n$ and find its stability group by demanding that

$$\sum_{l=1}^{3} \sum_{t=1}^{n} (\tilde{x}')^{-1}_{il} \delta_{lt} \tilde{o}''_{ts}(\gamma) = \delta_{is}.$$

The stability group is isomorphic to $SO(3) \times SO(n-3)$, where $SO(n-3)$ is a subgroup of SO(n) whereas SO(3) can be chosen to be either a subgroup of $\mathscr{GL}_+(3, \mathbb{R})$ or of SO(n). If we choose it to be the subgroup of $\mathscr{GL}_+(3, \mathbb{R})$ then we are again back to the $\mathscr{SO}(3) \times SO(n)$ scheme which we have already studied in § 2. We choose SO(3) to be a subgroup of SO(n). Then the orbit for the representative point given above is the coset space $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)/(SO(3) \times SO(n-3))$ and is parametrised by $3n$ real parameters. $SO(n)/(SO(3) \times SO(n-3))$ is the right coset space. In order to show that the orbit discussed so far is the only one under the $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ action, we make use of the $\mathcal{SO}(3)\times\mathrm{SO}(n)$ scheme. From this scheme we know that $\mathcal{GL}_+(3,\mathbb{R})$ may be decomposed as

$$\mathcal{GL}_+(3,\mathbb{R}) = \bigcup\{(\mathcal{SO}(3)/\mathcal{H}'_3)\times\mathring{x}\times\mathcal{SO}(3)\}$$

where $\mathcal{H}'_3$ is the stability group of the corresponding representative point $x_{is}=\delta_{is}\mathring{x}_i$. We insert this decomposition of $\mathcal{GL}_+(3,\mathbb{R})$ into the coset space $\mathcal{GL}_+(3,\mathbb{R})\times \mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$ and obtain $(\mathcal{SO}(3)/\mathcal{H}'_3)\times\mathring{x}\times\mathcal{SO}(3)\times\mathrm{SO}(n)/(\mathrm{SO}(3)\times \mathrm{SO}(n-3))$, which is equivalent to $(\mathcal{SO}(3)/\mathcal{H}'_3)\times\mathring{x}\times(\mathrm{SO}(n)/\mathrm{SO}(n-3))$. We are left with the manifold decomposition as in the $\mathcal{SO}(3)\times\mathrm{SO}(n)$ scheme which is complete. So we conclude.

Proposition 3.1.2. Under the action of $\mathcal{GL}_+(3,\mathbb{R})\times\mathrm{SO}(n)$, the $\mathbb{R}^{3n}$ obtains coordinates which are the parameters of the single orbit equivalent to the coset space $\mathcal{GL}_+(3,\mathbb{R})\times \mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$. We write for $x_{is}\in\mathbb{R}^{3n}$

$$x_{is}=\sum_{j=1}^{3} x'_{ij}o''_{js}(\beta) \qquad i=1,2,3, \qquad s=1,2,\ldots,n$$

where $x'\in\mathcal{GL}_+(3,\mathbb{R})$ and is parametrised by the real parameters $\mathcal{GL}_+(3,\mathbb{R})$ matrix elements $(x'_{ij}|i,j=1,2,3)$ and $o''(\beta)\in\mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$ is the right coset space. This decomposition is unique. The coordinates x'_{ij} can be considered as nine collective coordinates in the sense of $\mathcal{GL}_+(3,\mathbb{R})$ collectivity and the β are then $3n-9$ intrinsic coordinates. We define this coordinate transformation (CT) as ϕ_1:

$$\begin{aligned}(x'_{is}|i=1,2,3,\ s=1,2,\ldots,n) &\xrightarrow{\phi_1} (x'_{il},\beta_\tau|i,\ l=1,2,3,\ \tau=1,2,\ldots,3n-9)\\ &\equiv(\zeta_\nu|\nu=1,2,\ldots,3n).\end{aligned}$$

The CT ϕ_1 is only a first step: we would like to obtain some collective coordinates which we can interpret as the rotational ones, coordinates for vibrations and deformations and coordinates for vorticities. These coordinates can be easily introduced using the $\mathcal{SO}(3)\times\mathrm{SO}(n)$ scheme for $n=3$ already mentioned above, which provides us with the well known decomposition for the $\mathcal{GL}_+(3,\mathbb{R})$. So we introduce CT ϕ_2 and ϕ which is defined as $\phi:=\phi_2\circ\phi_1$.

Proposition 3.1.3. The coordinate transformation

$$(x_{is}|i=1,2,3,\ s=1,2,\ldots,n)\xrightarrow{\phi}(\omega_\mu,\mathring{x}_i,\omega'_{\mu'},\beta_\tau|\mu,\mu',i=1,2,3,\ \tau=1,2,\ldots,3n-9)$$

is defined by the unique decomposition of x_{is}:

$$x_{is}=\sum_{l=1}^{3}\sum_{k=1}^{3} o_{il}(\omega)\mathring{x}_l o'_{lk}(\omega')o''_{ks}(\beta)$$

where $o(\omega)\in\mathcal{SO}(3)/\mathcal{H}'_3$ is the left coset space and $o'(\omega')\in\mathcal{SO}(3)$, $o''(\beta)\in \mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$ is the right coset space. The CT ϕ we introduce by two successive CT $\phi_2\circ\phi_1$ where ϕ_1 is as in (3.1.2) and ϕ_2 is

$$\begin{aligned}&(x'_{jl},\beta_\tau|j,\ l=1,2,3,\ \tau=1,2,\ldots,3n-9)\xrightarrow{\phi_2}\\ &\qquad\times(\omega_\mu,\mathring{x}_i,\omega'_{\mu'},\beta_\tau|\mu,\mu',i=1,2,3,\ \tau=1,2,\ldots,3n-9)\end{aligned}$$

introduced through the unique decomposition of x':

$$x'_{ik} = \sum_{l=1}^{3} o_{il}(\omega)\mathring{x}_l o'_{lk}(\omega').$$

We restrict ourselves again, as in the $\mathscr{SO}(3)\times\mathrm{SO}(n)$ scheme, to the orbit of the type $\mathring{x}_1 > \mathring{x}_2 > \mathring{x}_3 > 0$ with $\mathscr{H}'_3 \equiv \mathrm{M}$ (M is the discrete group defined in § 2).

3.2. *The Jacobi matrix and its inverse for the coordinate transformations ϕ_1, ϕ_2 and ϕ*

In what follows we are not going to give the explicit inverse Jacobi matrix for the coordinate transformation ϕ. We will determine the Jacobi matrices and their inverse for ϕ_1 and ϕ_2. In all concrete calculations in (3.3)† and (3.4) we will proceed in two steps: first, find the expressions given after the coordinate transformation ϕ_1 using $\mathscr{J}^{-1}$ for ϕ_1 and then apply ϕ_2 using $\mathscr{J}^{-1}$ for ϕ_2 in order to determine the final form of the expressions studied in the coordinates related to ϕ.

Proposition 3.2.1. The Jacobi matrix for the coordinate transformation ϕ_1 introduced in (3.1.2):

$$(x_{is}|i=1,2,3,\ s=1,2,\ldots,n) \xrightarrow{\phi_1} (x'_{il}, \beta_\tau|i,\ l=1,2,3,\ \tau=1,2,\ldots,3n-9)$$
$$\equiv (\zeta_\nu|\nu=1,2,\ldots,3n)$$

is given through its matrix elements

$$\mathscr{J}_{\nu,is} \to \boxed{\begin{matrix} \mathscr{J}_{jl,is} \\[2ex] \mathscr{J}_{\tau,is} \end{matrix}} \equiv \boxed{\begin{matrix} \dfrac{\partial x_{is}}{\partial x'_{jl}} \\[2ex] \dfrac{\partial x_{is}}{\partial \beta_\tau} \end{matrix}}$$

(1) $$\frac{\partial x_{is}}{\partial x'_{jl}} = \delta_{ij} o''_{ls}(\beta)$$

(2) $$\frac{\partial x_{is}}{\partial \beta_\tau} = \sum_{m=1}^{3}\sum_{k=1}^{n} {}^{\mathrm{L}}\tilde{\Psi}_{\tau,mk}(\beta) x'_{im} o''_{ks}(\beta).$$

Proof. It follows trivially using (A18).

Proposition 3.2.2. The inverse Jacobi matrix $\mathscr{J}^{-1}$ for the coordinate transformation ϕ_1 introduced in (3.1.2) is given by its matrix elements

$$\mathscr{J}^{-1} \to \boxed{\mathscr{J}^{-1}_{is,jl} \qquad \mathscr{J}^{-1}_{is,\tau}} \equiv \boxed{\frac{\partial x'_{jl}}{\partial x_{is}} \qquad \frac{\partial \beta_\tau}{\partial x_{is}}}$$

† The symbol (3.3) means section 3, subsection 3.

$$(1)\quad \frac{\partial x'_{jl}}{\partial x_{is}} = \delta_{ij} o''_{ls}(\beta) + \sum_{m=1}^{3} \sum_{k=1}^{3} \sum_{m'=1}^{3} \sum_{k'=4}^{n} x'^{-1}_{m'i} o''_{k's}(\beta)\, {}^{\mathrm{L}}\Theta_{m'k',mk}(\beta) x'_{jm} \delta_{kl}$$

$$(2)\quad \frac{\partial \beta_\tau}{\partial x_{is}} = \sum_{m=1}^{3} \sum_{k=4}^{n} x'^{-1}_{mi} o''_{ks}(\beta)\, {}^{\mathrm{L}}\Theta_{mk,\tau}(\beta).$$

Proof. One makes the following ansatz

$$\mathcal{J}^{-1}_{\{is,{jl \atop \tau}\}} = \sum_{m=1}^{3} \sum_{k=1}^{n} \begin{array}{c|c} & m=1,2,3;\ k=1,2,\ldots,n \\ \hline i=1,2,3,\ s=1,2,\ldots,n & \boxed{x'^{-1}_{mi} o''_{ks}(\beta)} \end{array} \qquad \begin{array}{c|cc} & j,l=1,2,3 & \tau \\ \hline m,k=1,2,3 & B_{mk,jl} & 0 \\ m=1,2,3,\ k=4,5,\ldots,n & (\tilde{B}_1)_{mk,jl} & (\tilde{B}_2)_{mk,\tau} \end{array}$$

and using (A17) we derive that

$$B_{mk,jl} = x'_{jm} \delta_{kl}$$

$$(\tilde{B}_1)_{mk,jl} = \sum_{m'=1}^{3} \sum_{k'=1}^{3} {}^{\mathrm{L}}\Theta_{mk,m'k'}(\beta) x'_{jm'} \delta_{k'l}$$

and

$$(\tilde{B}_2)_{mk,\tau} = {}^{\mathrm{L}}\Theta_{mk,\tau}(\beta).$$

Proposition 3.2.3. The Jacobi matrix for the CT ϕ_2

$$(x'_{jl}, \beta\tau | j, l = 1,2,3,\ \tau = 1,2,\ldots,3n-9) \xrightarrow{\phi_2}$$
$$\times (\omega_\mu, \mathring{x}_i, \omega'_{\mu'}, \beta_\tau | \mu, \mu', i = 1,2,3,\ \tau = 1,2,\ldots,3n)$$

introduced in (3.1.3) is given through its non-trivial matrix elements identical to those obtained for the $\mathcal{SO}(3) \times \mathrm{SO}(n)$ scheme in (2.1) setting $n=3$, instead of x_{ls}, $l=1,2,3$, $s=1,2,\ldots,n$, x'_{jl}, $j, l = 1,2,3$ and instead of β_τ, $\omega'_{\mu'}$

$$(1)\qquad \frac{\partial x'_{jl}}{\partial \omega_\mu} = \sum_{i=1}^{3} \sum_{k=1}^{3} o_{jk}(\omega)\, {}^{\mathrm{R}}\tilde{\Psi}_{\mu,ki}(\omega) \mathring{x}_i o'_{il}(\omega')$$

$$(2)\qquad \frac{\partial x'_{jl}}{\partial \mathring{x}_w} = o_{jw}(\omega) o'_{wl}(\omega')$$

$$(3)\qquad \frac{\partial x'_{jl}}{\partial \omega'_{\mu'}} = \sum_{i=1}^{3} \sum_{k=1}^{3} o_{ji}(\omega) \mathring{x}_i\, {}^{\mathrm{L}}\tilde{\Psi}_{\mu',ik}(\omega') o'_{kl}(\omega').$$

Proposition 3.2.4. The inverse Jacobi matrix $\mathcal{J}^{-1}$ for CT ϕ_2 introduced in (3.1.3) is given through its non-trivial matrix elements identical to those obtained for the

$\mathcal{SO}(3)\times\mathrm{SO}(n)$ scheme in (2.2) setting $n=3$, instead of x_{ls}, $l=1,2,3$, $s=1,2,\ldots,n$, x'_{jl}, $j,l=1,2,3$, and instead of β_τ, $\omega'_{\mu'}$:

$$(1)\qquad \frac{\partial\omega_\mu}{\partial x'_{jl}}=\sum_{v=1}^{3}\sum_{g=1}^{3} o_{jg}(\omega)\frac{\mathring{x}_v}{\mathring{x}_v^2-\mathring{x}_g^2}\,{}^{\mathrm{R}}\tilde{\Theta}_{gv,\mu}(\omega)o'_{vl}(\omega')$$

$$(2)\qquad \frac{\partial\mathring{x}_w}{\partial x'_{jl}}=o_{jw}(\omega)o'_{wl}(\omega')$$

$$(3)\qquad \frac{\partial\omega'_{\mu'}}{\partial x'_{jl}}=\sum_{v=1}^{3}\sum_{g=1}^{3} o_{jg}(\omega)\frac{\mathring{x}_g}{\mathring{x}_g^2-\mathring{x}_v^2}\,{}^{\mathrm{L}}\tilde{\Theta}_{gv,\mu'}(\omega')o'_{vl}(\omega').$$

3.3. *The left and right action generators and other differential operators expressed through the new coordinates after the coordinate transformation ϕ_1 and ϕ_2*

In this subsection we consider only differential operators and we claim that the classical momenta can be obtained analogously.

Proposition 3.3.1. The right action generator of $\mathrm{SO}(n)$ acting on the manifold $\mathcal{GL}_+(3,\mathbb{R})\times\mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$, expressed through the new x' and β coordinates after the CT ϕ_1 given in (3.1.2), are

$${}^{\mathrm{R}}X^a_{st}=\sum_{m=1}^{3}\sum_{l=m+1}^{3}{}^{\mathrm{R}}\Theta_{st,ml}(\beta)\sum_{j=1}^{3}\left(x'_{jm}\frac{\partial}{\partial x'_{jl}}-x'_{jl}\frac{\partial}{\partial x'_{jm}}\right)+{}^{\mathrm{R}}L_{st}\qquad s<t=1,2,\ldots,n$$

where

$${}^{\mathrm{R}}L_{st}:=\sum_{\sigma=1}^{3n-9}{}^{\mathrm{R}}\Theta_{st,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma}.$$

After the successive ϕ_2 transformation, ${}^{\mathrm{R}}X^a_{st}$ takes the form

$${}^{\mathrm{R}}X^a_{st}=-\sum_{m=1}^{3}\sum_{l=m+1}^{3}\sum_{g=1}^{3}\sum_{v=g+1}^{3}{}^{\mathrm{R}}\Theta_{st,ml}(\beta)\,{}^{t}d_{ml,gv}(\omega')\tilde{\mathcal{L}}_{gv}+{}^{\mathrm{R}}L_{st}$$

where

$$\tilde{\mathcal{L}}_{gv}:=-\sum_{\mu'=1}^{3}{}^{\mathrm{L}}\Theta_{gv,\mu'}(\omega')\frac{\partial}{\partial\omega'_{\mu'}}\qquad g<v=1,2,3$$

is the vortex spin defined on the $\mathcal{SO}(3)$ manifold.

Proof. One uses the expression for $\mathcal{J}^{-1}$ for the CT ϕ_1 given in (3.2.2.), and then $\mathcal{J}^{-1}$ for CT ϕ_2 given in (3.2.4).

Proposition 3.3.2. The right action generators ${}^{\mathrm{R}}X^a_{st}$ of $\mathrm{SO}(n)$ given in (3.3.1) transcribed into the body-fixed frame are

$${}^{\mathrm{b}}X^a_{st}=\begin{cases}-\tilde{\mathcal{L}}_{st} & \text{for } s<t=1,2,3\\ -\sum_{m=1}^{3}\sum_{l=m+1}^{3}{}^{\mathrm{L}}\Theta^{b(\omega')}_{st,ml}(\beta)\tilde{\mathcal{L}}_{ml}+\tilde{\mathcal{I}}_{st} & \text{for } s=1,2,3,\ t=4,5,\ldots,n\\ 0 & \text{for } s<t=4,5,\ldots,n\end{cases}$$

where

$$^{\mathrm{L}}\Theta^{b(\omega')}_{st,ml}(\beta) := \sum_{k=1}^{3}\sum_{\tilde{m}-1}^{3}\sum_{\tilde{l}=\tilde{m}+1}^{3} o'_{sk}(\omega')\,^{\mathrm{L}}\Theta_{kt,\tilde{m}\tilde{l}}(\beta)\,^{\mathrm{t}}d_{\tilde{m}\tilde{l},ml}(\omega')$$

for $s = 1, 2, 3$, $t = 4, 5, \ldots, n$, $m < l = 1, 2, 3$ and

$$\tilde{\mathscr{I}}_{st} := \sum_{l=1}^{3} o'_{sl}(\omega')\,^{\mathrm{L}}\Theta_{lt,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma} \qquad \text{for } s = 1, 2, 3, t = 4, 5, \ldots, n.$$

Proof.

$$^{b}X^{a}_{st} := \begin{cases} \sum_{v=1}^{3}\sum_{g=v+1}^{3}\sum_{\tilde{s}=1}^{n}\sum_{\tilde{t}=\tilde{s}+1}^{n} {}^{\mathrm{t}}d^{-1}_{st,vg}(\omega')\,^{\mathrm{t}}d^{-1}_{vg,\tilde{s}\tilde{t}}(\beta)\,^{\mathrm{R}}X^{a}_{\tilde{s}\tilde{t}} & \text{for } s < t = 1, 2, 3 \\ \sum_{v=1}^{3}\sum_{\tilde{s}=1}^{n}\sum_{\tilde{t}=\tilde{s}+1}^{n} o'_{sv}(\omega')\,^{\mathrm{t}}d^{-1}_{vt,\tilde{s}\tilde{t}}(\beta)\,^{\mathrm{R}}X^{a}_{\tilde{s}\tilde{t}} & \text{for } s = 1, 2, 3, t = 4, 5, \ldots, n \\ 0 & \text{for } s < t = 4, 5, \ldots, n \end{cases}$$

and we use (A8) and (A17).

Proposition 3.3.3. The left action generators of $\mathscr{SO}(3)$ acting on the manifold $\mathscr{GL}_+(3, \mathbb{R}) \times \mathrm{SO}(n)/(\mathrm{SO}(3) \times \mathrm{SO}(n-3))$, expressed through the new coordinates after the coordinate transformation $\phi = \phi_2 \circ \phi_1$, are

$$^{\mathrm{L}}\mathscr{X}^{a}_{lk} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{\mathrm{t}}d^{-1}_{lk,gv}(\omega)L_{gv} \qquad l < k = 1, 2, 3$$

where

$$L_{gv} := \sum_{\mu=1}^{3} {}^{\mathrm{R}}\Theta_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \qquad g < 1, 2, 3.$$

Proof. After the CT ϕ_1 the left action $\mathscr{SO}(3)$ generators are defined on $\mathscr{GL}_+(3, \mathbb{R})$ manifold parametrised by x'_{ij}, and after the CT ϕ_2 one obtains the result in the same way as in (2.3) only for $n = 3$.

3.4. *The kinetic energy and the velocity field in $\mathscr{GL}_+(3, \mathbb{R}) \times SO(n)$ scheme*

Equipped with the inverse Jacobi matrix for the coordinate transformation ϕ_1 we calculate the complete kinetic energy for the relative motion of $n+1$ particles in new coordinates.

Proposition 3.4.1. The quantum mechanical kinetic energy on $\mathbb{R}^{3n}$ is given after the CT ϕ_1 (see (3.1.2)) in the new set of coordinates:

$$T \sim \sum_{j=1}^{3}\sum_{l=1}^{3}\frac{\partial^2}{\partial x'^2_{jl}} + \sum_{m=1}^{3}\sum_{m'=1}^{3}\sum_{k=4}^{n}\sum_{i=1}^{3} x'^{-1}_{mi}x'^{-1}_{m'i}\tilde{\mathscr{I}}_{m'k}\tilde{\mathscr{I}}_{mk}$$
$$+ \sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m=1}^{3}\sum_{m'=1}^{3}\sum_{\tilde{m}=1}^{3}\sum_{i=1}^{3} x'^{-1}_{mi}x'^{-1}_{m'i}\,^{\mathrm{L}}\Theta_{m'k,yl}(\beta)$$
$$\times\,^{\mathrm{L}}\Theta_{mk,\tilde{m}l'}(\beta)\sum_{j=1}^{3} x'_{jy}\frac{\partial}{\partial x'_{jl}}\sum_{j'=1}^{3} x'_{j'\tilde{m}}\frac{\partial}{\partial x'_{j'l'}}$$

$$-2\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3}\sum_{i=1}^{3} x'^{-1}_{mi}x'^{-1}_{m'i}\,{}^{\mathrm{L}}\Theta_{m'k,yl}(\beta)\sum_{j=1}^{3}x'_{jy}\frac{\partial}{\partial x'_{jl}}\tilde{\mathscr{I}}_{mk}$$

$$+\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3}\sum_{\tilde{m}=1}^{3}{}^{\mathrm{L}}\Theta_{m'k,yl}(\beta)$$

$$\times{}^{\mathrm{L}}\Theta_{mk,\tilde{m}l'}(\beta)x'^{-1}_{mi}\sum_{j=1}^{3}x'_{jy}\left(\frac{\partial}{\partial x'_{jl}}x'^{-1}_{mi}\right)\sum_{j'=1}^{3}x'_{j'\tilde{m}}\frac{\partial}{\partial x'_{jl'}}$$

$$-\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3}{}^{\mathrm{L}}\Theta_{m'k,yl}(\beta)x'^{-1}_{m'i}\sum_{j=1}^{3}x'_{jy}\left(\frac{\partial}{\partial x'_{jl}}x'^{-1}_{mi}\right)\tilde{\mathscr{I}}_{mk}$$

$$-\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{l'=1}^{3}\sum_{m'=1}^{3}\sum_{\tilde{m}=1}^{3}\sum_{i=1}^{3}x'^{-1}_{mi}x'^{-1}_{m'i}[\tilde{\mathscr{I}}_{m'k}\,{}^{\mathrm{L}}\Theta_{mk,\tilde{m}l'}(\beta)]\sum_{j=1}^{3}x'_{j\tilde{m}}\frac{\partial}{\partial x'_{jl'}}$$

$$-\sum_{m'=1}^{3}\sum_{k=4}^{n}\sum_{k'=4}^{n}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{\tilde{m}=1}^{3}\sum_{i=1}^{3}x'^{-1}_{mi}x'^{-1}_{m'i}\,{}^{\mathrm{L}}\Theta_{m'k,k'k}(\beta)$$

$$\times{}^{\mathrm{L}}\Theta_{mk',\tilde{m}l'}(\beta)\sum_{j=1}^{3}x'_{j\tilde{m}}\frac{\partial}{\partial x'_{jl'}}$$

$$+\sum_{m=1}^{3}\sum_{m'=1}^{3}\sum_{k=4}^{n}\sum_{k'=4}^{n}\sum_{i=1}^{3}x'^{-1}_{mi}x'^{-1}_{m'i}\,{}^{\mathrm{L}}\Theta_{m'k,k'k}(\beta)\tilde{\mathscr{I}}_{mk'}$$

where

$$\tilde{\mathscr{I}}_{mk}:=-\sum_{\tau=1}^{3n-9}{}^{\mathrm{L}}\Theta_{mk,\tau}(\beta)\frac{\partial}{\partial\beta\tau}\qquad m<k=1,2,\ldots,n$$

and

$${}^{\mathrm{L}}\Theta_{m'<k',m<k}(\beta):=\sum_{\mu=1}^{n(n-1)/2}{}^{\mathrm{L}}\Theta_{m'<k',\mu}(\beta)P_{\mu,m<k}$$

$$P_{\mu,m<k}:=\delta_{\mu,m<k}\qquad m,k,m',k'=1,2,\ldots,n.$$

We are not going to discuss the rather complicated expression obtained in (3.4.1) because we use it only as an intermediate result. What we are really interested in is the question of whether the kinetic energy in new coordinates after the CT ϕ separates into $\mathscr{GL}_+(3,\mathbb{R})$ collective and intrinsic parts. For this goal it is only important to calculate all the quadratic terms in momenta in the expression for the kinetic energy given in (3.4.1) after the CT ϕ_2. This can easily be done making use of (3.2.4).

Proposition 3.4.2. The quantum mechanical expression for the kinetic energy quadratic in momenta ($T_{(Q)}$) under the CT defined in (3.1.3)

$$(x_{is}|i=1,2,3,\ s=1,2,\ldots,n)\xrightarrow{\phi}(\omega_\mu,\ \mathring{x}_i,\ \omega'_{\mu'},\ \beta_\tau|\mu,\ i,\ \mu'=1,2,3,\ \tau=1,2,\ldots,3n-9)$$

takes the form

$$T_{(Q)}=\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\mathring{x}_g^2+\mathring{x}_v^2}{(\mathring{x}_g^2-\mathring{x}_v^2)^2}\hat{L}^2_{gv}+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\mathring{x}_g^2+\mathring{x}_v^2}{(\mathring{x}_g^2-\mathring{x}_v^2)^2}\hat{\hat{\mathscr{L}}}^2_{gv}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{4\mathring{x}_g\mathring{x}_v}{(\mathring{x}_g^2-\mathring{x}_v^2)^2}\hat{L}_{gv}\hat{\hat{\mathscr{L}}}_{gv}+\frac{1}{2m}\sum_{v=1}^{3}\hat{t}_v^2+\frac{1}{2m}\sum_{k=4}^{n}\sum_{i=1}^{3}\mathring{x}_i^{-2}\hat{\hat{\mathscr{I}}}^2_{ik}$$

$$+\frac{1}{2m}\sum_{k=4}^{n}\sum_{i=1}^{3}\sum_{g=1}^{3}\sum_{v=g+1}^{3} 2\mathring{x}_i^{-2}\,{}^{\mathrm{L}}\Theta_{ik,gv}^{b(\omega')}(\beta)\hat{\hat{\mathscr{L}}}_{gv}\hat{\hat{\mathscr{I}}}_{ik}$$

$$+\frac{1}{2m}\sum_{k=4}^{n}\sum_{i=1}^{3}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\sum_{\tilde{g}=1}^{3}\sum_{\tilde{v}=\tilde{g}+1}^{3} \mathring{x}_i^{-2}\,{}^{\mathrm{L}}\Theta_{ik,gv}^{b(\omega')}(\beta)\,{}^{\mathrm{L}}\Theta_{ik,\tilde{g}\tilde{v}}^{b(\omega')}(\beta)\hat{\hat{\mathscr{L}}}_{gv}\hat{\hat{\mathscr{L}}}_{\tilde{g}\tilde{v}}$$

where

$$\hat{L}_{gv} := -\mathrm{i}\hbar\sum_{\mu=1}^{3}{}^{\mathrm{R}}\Theta_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \qquad g<v=1,2,3$$

$$\hat{t}_v := -\mathrm{i}\hbar\frac{\partial}{\partial\mathring{x}_v} \qquad v=1,2,3$$

$$\hat{\hat{\mathscr{L}}}_{gv} := \mathrm{i}\hbar\sum_{\mu'=1}^{3}{}^{\mathrm{L}}\Theta_{gv,\mu'}(\omega')\frac{\partial}{\partial\omega'_{\mu'}} \qquad g<v=1,2,3$$

$$\hat{\hat{\mathscr{I}}}_{ik} := \mathrm{i}\hbar\sum_{m=1}^{3}\sigma'_{im}(\omega')\sum_{\tau=1}^{3n-9}{}^{\mathrm{L}}\Theta_{mk,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad i=1,2,3,\ k=4,5,\ldots,n$$

$${}^{\mathrm{L}}\Theta_{ik,gv}^{b(\omega')}(\beta) := \sum_{m=1}^{3}\sum_{l=1}^{3}\sum_{a=1}^{3}\sigma'_{im}(\omega')\sigma'_{gl}(\omega')\sigma'_{va}(\omega')\,{}^{\mathrm{L}}\tilde{\Theta}_{mk,la}(\beta)$$

$$i=1,2,3, \qquad g<v=1,2,3, \qquad k=4,5,\ldots,n.$$

Let us discuss the expression obtained in (3.4.2). Through the CT ϕ we defined the vortex spin $\hat{\hat{\mathscr{L}}}$ on an $\mathscr{SO}(3)$ manifold. The intrinsic momenta $\hat{\hat{\mathscr{I}}}$ contain no partial derivatives WRT the $\mathscr{GL}_+(3,\mathbb{R})$ parameters, or more exactly, they do not contain $\mathscr{SO}(3)$ parameters which define the vortex spin $\hat{\hat{\mathscr{L}}}$. All this can be seen from (3.3.1). It is not contained in the $\mathscr{SO}(3)\times\mathrm{SO}(n)$ scheme, where both vortex spin $\hat{\hat{\mathscr{L}}}$ and the intrinsic operators $\hat{\mathscr{I}}$ were defined on the right coset $\mathrm{SO}(n)/\mathrm{SO}(n-3)$ (see (2.4) and compare it with (3.3.1) and (3.3.2)). From (3.4.2) it becomes clear that a complete separation has not been achieved: in the sixth and seventh terms in $T_{(Q)}$ coupling terms appear. The matrix block different from zero which causes those terms is ${}^{\mathrm{L}}\Theta_{mk,li}^{b(\omega')}(\beta)$ or equivalently ${}^{\mathrm{L}}\Theta_{mk,li}(\beta)$ for $m=1,2,3$, $k=4,5,\ldots,n$, $l<i=1,2,3$ (see (3.3.2)). Only if we postulate that ${}^{\mathrm{L}}\Theta_{mk,li}(\beta)=0$ would we obtain the complete separation. This condition can be restated in terms of the matrix ${}^{\mathrm{L}}\Psi$, inverse to ${}^{\mathrm{L}}\Theta$, and the condition becomes ${}^{\mathrm{L}}\Psi_{mk,li}(\beta)=0$ for the same range of indices (see (A3) and (A17)).

Now we compare our results with the approach taken by Gulshani and Rowe (1976). The constraint functions introduced by these authors may be written in our notation as

$$\sum_{s=1}^{n}\left(\sigma''_{bs}(\beta)\frac{\partial\sigma''_{vs}(\beta)}{\partial\beta_\tau}-\sigma''_{vs}(\beta)\frac{\partial\sigma''_{bs}(\beta)}{\partial\beta_\tau}\right)=2\,{}^{\mathrm{L}}\tilde{\Psi}_{\tau,bv}$$

where b, $v=1,2,3$ (see (A19)). Actually σ'' of these authors is some $3\times n$ matrix which in the present analysis can be identified as $\sigma''\in\mathrm{SO}(n)/(\mathrm{SO}(3)\times\mathrm{SO}(n-3))$. The authors in equation (2.9) require the vanishing of the constraint functions and hence formulate three non-holonomic constraints from altogether nine constraints on σ''_{bs} there denoted by $x''_{n\alpha}$, $n=1,2,\ldots,N(N\equiv A-1)$ and $\alpha=1,2,3$. These constraints define $3n-9$ intrinsic coordinates β_τ denoted there by ξ_σ. The constraint functions by our analysis are identified as part of the matrix ${}^{\mathrm{L}}\Psi$ considered above. So we have shown that the separation of the kinetic energy enforces the non-holonomic constraints.

Now we try to define the linear (collective) velocity field. Making use of $\mathscr{J}^{-1}$ for both CT ϕ_1 and ϕ_2 we obtain the candidate for p_{is}^{coll}:

$$p_{is}^{\mathrm{coll}}=\sum_{l=1}^{3}\chi_{il}^{S}x_{ls}+\sum_{l=1}^{3}\chi_{il}^{A}x_{ls}$$
$$-\sum_{l=1}^{3}\sum_{k=4}^{n}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\mathring{x}_l^{-1}o_{il}(\omega)\,{}^{\mathrm{L}}\Theta_{lk,gv}^{b(\omega')}(\beta)o'_{ks}(\beta)\tilde{\mathscr{L}}_{gv}$$

where

$$\chi_{il}^{S}:=\sum_{g=1}^{3}\sum_{v=g+1}^{n}(o_{ig}(\omega)o_{lv}(\omega)+o_{lg}(\omega)o_{iv}(\omega))\frac{1}{\mathring{x}_v^2-\mathring{x}_g^2}\left(L_{gv}+\frac{\mathring{x}_g^2+\mathring{x}_v^2}{2\mathring{x}_v\mathring{x}_g}\tilde{\mathscr{L}}_{gv}\right)$$
$$+\sum_{w=1}^{3}o_{iw}(\omega)\mathring{x}_w^{-1}o_{lw}(\omega)\pi_w$$
$$\chi_{il}^{S}=\chi_{li}^{S}$$

and

$$\chi_{il}^{A}:=\sum_{g=1}^{3}\sum_{v=g+1}^{3}(o_{iv}(\omega)o_{lg}(\omega)-o_{lv}(\omega)o_{ig}(\omega))\frac{1}{2\mathring{x}_v\mathring{x}_g}\tilde{\mathscr{L}}_{gv}$$
$$\chi_{il}^{A}=-\chi_{li}^{A}$$

and where L_{gv}, $\tilde{\mathscr{L}}_{gv}$ and π_w are the classical momenta which take the same form as the corresponding differential operators with partial derivatives replaced by the canonical momenta. The last line in p_{is}^{coll} proportional to ${}^{\mathrm{L}}\Theta_{mk,li}(\beta)$, $m=1,2,3$, $k=4,5,\ldots,n$, $l<i=1,2,3$, prevents the separation.

4. Conclusions

We derived the kinetic energy with an exact expression for the inverse of the Jacobi matrix $\mathscr{J}^{-1}$ for CT ϕ based on the orbit analysis WRT the $\mathscr{GL}_+(3,\mathbb{R})\times\mathrm{SO}(n)$ group. We conclude that the constraints ${}^{\mathrm{L}}\Theta_{mk,li}(\beta)=0$ for $m=1,2,3$, $k=4,5,\ldots,n$, $l<i=1,2,3$ are necessary and sufficient if one wants to obtain the total separation of kinetic energy into intrinsic and collective parts WRT the $\mathscr{GL}_+(3,\mathbb{R})$ collectivity. The same holds true for the separation of the velocity field. It should be stressed that the constraints and hence the separation do not follow from the theory of the Lie groups involved.

Appendix. The action of a Lie group on its parameter space

In this appendix we give a set of definitions and propositions which we use as the tools in our calculations. Some of them are well known but we give them for completeness. Each proposition is given without proof; instead, at the end of each one we indicate the propositions and definitions we used in order to prove it.

We study first the left action of a group $\tilde{\mathrm{G}}$ on its parameter space $\mathrm{G}(\mathrm{G}\simeq\tilde{\mathrm{G}})$ and assume everywhere the exponential parametrisation of G or $\tilde{\mathrm{G}}$ which implies that

$$g^{-1}(\gamma)=g(\gamma^{-1})=g(-\gamma)\qquad g\in\mathrm{G}.$$

The left action of $\tilde{G}$ on G is globally given by

$$\tilde{G} \times G \to G$$
$$\tilde{g}^{-1} g = g' \qquad \text{where } \tilde{g} \in \tilde{G},\ g, g' \in G.$$

Definition A1. Given the structure function $\phi_\mu(\gamma, \chi)$ of the group G, the matrix ${}^{L}\Theta(\chi)$ can be introduced:

$${}^{L}\Theta_{\rho\mu}(\chi) := \frac{\partial}{\partial \tilde{\gamma}_\rho} \phi_\mu(\tilde{\gamma}, \chi)\bigg|_{\tilde{\gamma}=0}.$$

Definition A2. The generators for the left action

$$\tilde{G} \times G \to G$$

are

$${}^{L}X_\rho(\chi) = -\sum_\mu {}^{L}\Theta_{\rho\mu}(\chi) \frac{\partial}{\partial \chi_\mu}.$$

Definition A3. The matrix ${}^{L}\Psi(\chi)$ (Gilmore 1974, p 97, relation (2.6)) is introduced as the inverse of ${}^{L}\Theta(\chi)$:

$${}^{L}\Psi\, {}^{L}\Theta = {}^{L}\Theta\, {}^{L}\Psi = \mathbb{I}.$$

Now we present the equivalent expressions for the right action $G \times \tilde{G} \to G$ and its relation to the left action.

Definition A4. Given the structure function $\phi_\mu(\chi, \gamma)$, the matrix ${}^{R}\Theta(\chi)$ can be introduced:

$${}^{R}\Theta_{\rho\mu}(\chi) := \frac{\partial}{\partial \tilde{\gamma}_\rho} \phi_\mu(\chi, \tilde{\gamma})\bigg|_{\tilde{\gamma}=0}.$$

Definition A5. The generators for the right action $G \times \tilde{G} \to G$ are

$${}^{R}X_\rho(\chi) = \sum_\mu {}^{R}\Theta_{\rho\mu}(\chi) \frac{\partial}{\partial \chi_\mu}.$$

Definition A6. The matrix ${}^{R}\Psi(\chi)$ is introduced as the inverse of ${}^{R}\Theta(\chi)$.

Proposition A7. The matrices $d(\chi)$ defined as

$${}^{t}d_{\mu\rho}(\chi) = \frac{\partial \phi_\rho(\phi(\chi, \gamma), \chi^{-1})}{\partial \gamma_\mu}\bigg|_{\gamma=0}$$

form a representation of G called the adjoint representation (Ad representation).

Proposition A8. The right and left action generators are related linearly through the adjoint representation

$${}^{R}X_\mu(\chi) = -\sum_\sigma {}^{t}d_{\mu\sigma}(\chi)\, {}^{L}X_\sigma(\chi)$$

$$^{\mathrm{R}}\Theta_{\mu\lambda}(\chi)=\sum_{\sigma}{}^{\mathrm{t}}d_{\mu\sigma}(\chi)\,{}^{\mathrm{L}}\Theta_{\sigma\lambda}(\chi).$$

(A7), (A1), (A2).

Proposition A9. The matrix $^{\mathrm{R}}\Psi(\chi)$ defined in (A6) is given as

$$^{\mathrm{R}}\Psi_{\rho\nu}(\chi)=\frac{\partial}{\partial\tilde{\gamma}_{\rho}}\phi_{\nu}(\chi^{-1},\tilde{\gamma})\bigg|_{\tilde{\gamma}=\chi}$$

(A4).

Proposition A10. The matrices $^{\mathrm{L}}\Psi(\chi)$ and $^{\mathrm{R}}\Psi(\chi)$ are linearly related through the adjoint representation

$$^{\mathrm{L}}\Psi(\chi)={}^{\mathrm{R}}\Psi(\chi)^{\mathrm{t}}d(\chi)$$

(A6), (A8).

Proposition A11. The matrix $^{\mathrm{L}}\Psi(\chi)$ defined in (A3) is given as

$$^{\mathrm{L}}\Psi_{\nu\rho}(\chi)=\frac{\partial}{\partial\beta_{\nu}}\phi_{\rho}(\beta,\chi^{-1})\bigg|_{\beta=\chi}$$

(A7), (A9), (A10).

Proposition A12. The following relation is satisfied:

$$\frac{\partial}{\partial\gamma_{\beta}}{}^{\mathrm{t}}d_{\alpha\sigma}(\gamma)\bigg|_{\gamma=0}=C^{\sigma}_{\beta\alpha}$$

where $d(\gamma)$ is the adjoint representation and the $C^{\sigma}_{\beta\alpha}$ are structure constants (Gilmore 1974, pp 102, 103, relations (3.14), (3.23))

$$[{}^{\mathrm{L}}X_{\alpha},{}^{\mathrm{L}}X_{\beta}]=\sum_{\rho}C^{\rho}_{\alpha\beta}\,{}^{\mathrm{L}}X_{\rho}.$$

(A4), (A7), (A11).

Proposition A13. The left and right action generators commute among each other:

$$[{}^{\mathrm{L}}X_{\mu},{}^{\mathrm{R}}X_{\nu}]=0\qquad\forall\mu,\nu.$$

Propositions A14. The right action generators fulfil the commutation relations with the same structure constants as the left action generators:

$$[{}^{\mathrm{R}}X_{\alpha},{}^{\mathrm{R}}X_{\beta}]=\sum_{\rho}C^{\rho}_{\alpha\beta}\,{}^{\mathrm{R}}X_{\rho}.$$

(A8), (A12), (A13).

Now we choose the parameters of the group $\tilde{\mathrm{G}}$ and G ($\tilde{\mathrm{G}}\simeq\mathrm{G}$) so that certain subsets of them parametrise a subgroup H of G, $\mathrm{H}<\mathrm{G}$. The group manifold G can be written as $\mathrm{G}=\mathrm{H}C^{\mathrm{R}}$ where $C^{\mathrm{R}}=\mathrm{G}/\mathrm{H}$ is the right coset space. The parameters of the group G are denoted by γ_{μ}, $\gamma_{\mu'}$ and determine $g(\gamma)\in\mathrm{G}$; the parameters of the subgroup H are $\delta_{\kappa}, \delta_{\kappa'}, \ldots, \delta_{\lambda}, \delta_{\lambda'}, \ldots$ and determine $h(\delta)\in\mathrm{H}$, the parameters of the right coset space C^{R} are $\beta_{\tau}, \beta_{\tau'}, \ldots, \beta_{\sigma}, \beta_{\sigma'}, \ldots$ and determine $c(\beta)\in C^{\mathrm{R}}$. In what follows, the indices $\kappa, \kappa', \ldots, \lambda, \lambda', \ldots$ will always refer to the subgroup parameters δ. The indices $\tau, \tau', \ldots$, and the $\sigma, \sigma', \ldots$, refer to the right coset parameters β, and the indices $\mu, \mu' \ldots$, refer to the parameters γ of the full group.

We start with the right action, $\mathrm{H}C^{\mathrm{R}} \times \mathrm{G} \to \mathrm{H}C^{\mathrm{R}}$. The right action generators can be written

$$ {}^{\mathrm{R}}X_{\mu} = \sum_{\lambda} {}^{\mathrm{R}}\Theta_{\mu\lambda}(\gamma) \frac{\partial}{\partial \delta_{\lambda}} + \sum_{\tau} {}^{\mathrm{R}}\Theta_{\mu\tau}(\gamma) \frac{\partial}{\partial \beta_{\tau}}. $$

It can easily be shown that ${}^{\mathrm{R}}\Theta_{\mu\tau}(\gamma) = {}^{\mathrm{R}}\Theta_{\mu\tau}(\beta)$ and analogously ${}^{\mathrm{R}}\Psi_{\sigma\mu}(\gamma) = {}^{\mathrm{R}}\Psi_{\sigma\mu}(\beta)$. We are particularly interested in the right action of the group $\tilde{\mathrm{G}}$ on some right coset space $C^{\mathrm{R}} = \mathrm{G}/\mathrm{H}$. That such an action is well defined is evident from the global argument

$$ g(\gamma) = h(\delta) c(\beta) \qquad g\tilde{g} = hc\tilde{g} = hh'c' $$

where $c' \neq c'(\delta)$ and is equivalent to $c(\beta)\tilde{g} \to c'(\beta')$. So we write the following well known proposition.

Proposition A15. The manifold which supports the right group action $\tilde{\mathrm{G}}$ may be chosen to be the right coset space C^{R} so that $C^{\mathrm{R}} \times \tilde{\mathrm{G}} \to C^{\mathrm{R}}$. The right action generators are

$$ {}^{\mathrm{R}}X_{\mu}(\beta) = \sum_{\tau} {}^{\mathrm{R}}\Theta_{\mu\tau}(\beta) \frac{\partial}{\partial \beta_{\tau}} $$

and the corresponding ${}^{\mathrm{R}}\Psi(\beta)$ and ${}^{\mathrm{R}}\Theta(\beta)$ matrices are related by

$$ \sum_{\mu} {}^{R}\Psi_{\sigma\mu}(\beta)\, {}^{\mathrm{R}}\Theta_{\mu\tau}(\beta) = \delta_{\sigma\tau}. $$

Proposition A16. Within the new notation, the matrices ${}^{\mathrm{L}}\Theta(\delta\beta)$ and ${}^{\mathrm{L}}\Psi(\delta\beta)$ take the following block form:

$$ {}^{\mathrm{L}}\Theta(\delta\beta) \to \begin{array}{c c} & \begin{array}{cc} \quad\lambda\quad & \quad\tau\quad \end{array} \\ \begin{array}{c} \kappa \\ \sigma \end{array} & \left[\begin{array}{c|c} {}^{\mathrm{L}}\Theta_{\kappa\lambda}(\delta) & 0 \\ \hline \left.\dfrac{\partial}{\partial \tilde{\beta}_{\sigma}} \phi_{\lambda}(\tilde{\beta}, \bar{\partial}\beta)\right|_{\tilde{\beta}=0} & \left.\dfrac{\partial}{\partial \tilde{\beta}_{\sigma}} \phi_{\tau}(\tilde{\beta}, \delta\beta)\right|_{\tilde{\beta}=0} \end{array}\right] \end{array} $$

$$ {}^{\mathrm{L}}\Psi(\delta\beta) \to \begin{array}{c c} & \begin{array}{cc} \quad\lambda\quad & \quad\tau\quad \end{array} \\ \begin{array}{c} \kappa \\ \sigma \end{array} & \left[\begin{array}{c|c} {}^{\mathrm{L}}\Psi_{\kappa\lambda}(\delta) & 0 \\ \hline \left.\dfrac{\partial}{\partial \tilde{\beta}_{\sigma}} \phi_{\lambda}(\delta\tilde{\beta}, \beta^{-1}\delta^{-1})\right|_{\tilde{\beta}=\beta} & \left.\dfrac{\partial}{\partial \tilde{\beta}_{\sigma}} \phi_{\tau}(\tilde{\beta}, \beta^{-1}\delta^{-1})\right|_{\tilde{\beta}=\beta} \end{array}\right] \end{array} $$

(A1), (A11).

Proposition A17. The matrix blocks from the matrics ${}^{\mathrm{L}}\Theta(\delta\beta)$ and ${}^{\mathrm{L}}\Psi(\delta\beta)$ given in (A16) are related in the following way to each other:

$$ \sum_{\lambda'} {}^{\mathrm{L}}\Psi_{\kappa\lambda'}(\delta)\, {}^{\mathrm{L}}\Theta_{\lambda'\lambda}(\delta) = \sum_{\lambda'} {}^{\mathrm{L}}\Theta_{\kappa\lambda'}(\delta)\, {}^{\mathrm{L}}\Psi_{\lambda'\lambda}(\delta) = \delta_{\kappa\lambda} $$

$$ \sum_{\tau'} {}^{\mathrm{L}}\Psi_{\sigma\tau'}(\delta\beta)\, {}^{\mathrm{L}}\Theta_{\tau'\tau}(\delta\beta) = \sum_{\tau'} {}^{\mathrm{L}}\Theta_{\sigma\tau'}(\delta\beta)\, {}^{\mathrm{L}}\Psi_{\tau'\tau}(\delta\beta) = \delta_{\sigma\tau} $$

$$\sum_{\lambda} {}^{\mathrm{L}}\Theta_{\sigma\lambda}(\delta\beta)\,{}^{\mathrm{L}}\Psi_{\lambda\kappa}(\delta)+\sum_{\tau} {}^{\mathrm{L}}\Theta_{\sigma\tau}(\delta\beta)\,{}^{\mathrm{L}}\Psi_{\tau\kappa}(\delta\beta)=0.$$

Setting $\delta = e(0)$, the last two relations become

$$\sum_{\tau'} {}^{\mathrm{L}}\Psi_{\sigma\tau'}(\beta)\,{}^{\mathrm{L}}\Theta_{\tau'\tau}(\beta)=\sum_{\tau'} {}^{\mathrm{L}}\Theta_{\sigma\tau'}(\beta)\,{}^{\mathrm{L}}\Psi_{\tau'\tau}(\beta)=\delta_{\sigma\tau}$$

$${}^{\mathrm{L}}\Theta_{\sigma\kappa}(\beta)+\sum_{\tau} {}^{\mathrm{L}}\Theta_{\sigma\tau}(\beta)\,{}^{\mathrm{L}}\Psi_{\tau\kappa}(\beta)=0.$$

(A3), (A17).

One should note that although some left action of G on the right coset space cannot be defined in general, the matrices ${}^{\mathrm{L}}\Theta(\beta)$ and ${}^{\mathrm{L}}\Psi(\beta)$ can be defined.

Now we restrict our analysis to an orthogonal SO(j) matrix group, i.e. on (100 . . .) (Hamermesh 1964, pp 396, 397), the fundamental representative of SO(j) which is exponentially parametrised. The infinitesimal generator basis consists of $L_{i<l} := c_{il} - c_{li}$ for $i, l = 1, 2, \ldots, j$ and $(c_{il})_{kj} = \delta_{ik}\delta_{lj}$. Any element of the Lie algebra can be written

$$X = \sum_{i<l} \varphi_{i<l} L_{i<l}$$

with real parameters $\varphi_{i<l}$. In those places where we are not much interested in details, we write φ_μ instead of $\varphi_{i<l}$.

Propositions A18.

$$\sum_{k=1}^{j}\left(\frac{\partial}{\partial\varphi_\mu}o_{ik}(\varphi)\right){}^{\mathrm{t}}o_{kl}(\varphi) := {}^{\mathrm{L}}\tilde{\Psi}_{\mu,il}(\varphi)$$

$$= \begin{cases} {}^{\mathrm{L}}\Psi_{\mu,i<l} & \text{for } i<l \\ 0 & \text{for } i=l \\ -{}^{\mathrm{L}}\Psi_{\mu,l<i} & \text{for } i>l \end{cases}$$

(A.11).

Proposition A19. The adjoint representation $d(\varphi)$ can be written

$$o_{ik}(\varphi)\,{}^{\mathrm{t}}o_{ml}(\varphi) - o_{im}(\varphi)\,{}^{\mathrm{t}}o_{kl}(\varphi) = \begin{cases} {}^{\mathrm{t}}d_{km,il}(\varphi) & \text{for } k<m,\ i<l \\ -{}^{\mathrm{t}}d_{km,li}(\varphi) & \text{for } k<m,\ i>l \\ -{}^{\mathrm{t}}d_{mk,il}(\varphi) & \text{for } k>m,\ i<l \\ {}^{\mathrm{t}}d_{mk,li}(\varphi) & \text{for } k>m,\ i>l \\ 0 & \text{for all other cases} \end{cases}$$

(A7).

Proposition A20.

$${}^{\mathrm{R}}\Psi_{\mu,i<l}(\varphi) = \sum_{k=1}^{j} {}^{\mathrm{t}}o_{ik}(\varphi)\frac{\partial}{\partial\varphi_\mu}o_{kl}(\varphi).$$

(A10), (A18), (A19).

Proposition A21.

$$\sum_{k=1}^{j} {}^{t}o_{ik}(\varphi)\frac{\partial}{\partial\varphi_\mu}o_{kl}(\varphi) =: {}^{R}\tilde{\Psi}_{\mu,il}(\varphi)$$

$$= \begin{cases} {}^{R}\Psi_{\mu,il} & \text{for } i<l \\ 0 & \text{for } i=l \\ -{}^{R}\Psi_{\mu,li} & \text{for } i>l. \end{cases}$$

(A11).

Proposition A22. $\tilde{\Psi}_{\mu,il} = -\tilde{\Psi}_{\mu,li}$ both for the left and right action (A18), (A21).

Definition A23.

$$\tilde{\Theta}_{il,\mu} := \begin{cases} \Theta_{il,\mu} & \text{for } i<l \\ 0 & \text{for } i=l \\ -\Theta_{li,\mu} & \text{for } i>l \end{cases}$$

both for the left and right action.

Proposition A24. $\tilde{\Theta}_{il,\mu} = -\tilde{\Theta}_{li,\mu}$ both for the left and right action (A23).

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